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ON THE CALCULATION OF THE TEMPERATURE FIELD IN SOLIDS WITH VARIABLE HEAT-TRANSFER COEFFICIENTS

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A method is presented for the approximate determination of the temperature field in solids with variable heat-transfer coefficients.

The mathematical formulation of the problem of the temperature field in a solid body with a variable heat-transfer coefficient is:

$$\frac{\partial\vartheta(\Psi,\ \mathrm{Fo})}{\partial\,\mathrm{Fo}}=\frac{\partial^2\vartheta(\Psi,\ \mathrm{Fo})}{\partial\,\Psi^2}+\frac{K-1}{\Psi}\ \frac{\partial\vartheta(\Psi,\ \mathrm{Fo})}{\partial\Psi}, \tag{1}$$

$$\partial\vartheta(1,\ \mathrm{Fo})/\partial\Psi=\mathrm{Bi}(\mathrm{Fo})\,[1-\vartheta(1,\ \mathrm{Fo})], \tag{2}$$

$$\partial\vartheta(0,\ \mathrm{Fo})/\partial\Psi=0, \tag{3}$$

where

$$\vartheta(\Psi,\ 0)=\vartheta_0, \tag{4}$$

$$\vartheta=T/T_c,\ \ 0\leqslant\Psi\leqslant 1,\ \ 0\leqslant \mathrm{Fo}=a\tau/R^2<\infty\ ,$$

$\mathrm{Bi} = \alpha R/\lambda$, and the factor K is equal to 1, 2, or 3 in the plane, cylindrical, and spherical problem, respectively.

Introducing a new variable $\xi(\Psi,\ \mathrm{Fo})$, related to $\vartheta(\Psi,\ \mathrm{Fo})$ by

$$\xi(\Psi,\ \mathrm{Fo})=\ln[1-\vartheta(\Psi,\ \mathrm{Fo})], \tag{5}$$

we may rewrite the system (1)-(4) in the form:

$$\frac{\partial\xi(\Psi,\ \mathrm{Fo})}{\partial\,\mathrm{Fo}}=\frac{\partial^2\xi(\Psi,\ \mathrm{Fo})}{\partial\Psi^2}+\frac{K-1}{\Psi}\ \frac{\partial\xi(\Psi,\ \mathrm{Fo})}{\partial\Psi}+\left[\frac{\partial\xi(\Psi,\ \mathrm{Fo})}{\partial\Psi}\right]^2, \tag{6}$$

$$\partial\xi(1,\ \mathrm{Fo})/\partial\Psi=-\mathrm{Bi}(\mathrm{Fo}), \tag{7}$$

$$\partial\xi(0,\ \mathrm{Fo})/\partial\Psi=0, \tag{8}$$

$$\xi(\Psi,\ 0)=\ln(1-\vartheta_0). \tag{9}$$

The quantity $(\partial\xi/\partial\Psi)^2$ in (6) increases monotonically from zero at $\Psi = 0$ to a maximum value at $\Psi = 1$, and can be regarded from the physical point of view as an internal heat source of variable strength. Being a function of Fo and Bi, this quantity decreases with decreasing Bi and becomes negligibly small for $\mathrm{Bi} \le 0.25$ ("thin" body). Assuming $(\partial\xi/\partial\Psi)^2 = 0$, using the solutions of the system (6)-(9), and taking account of (5), we obtain the required temperature distribution.

Infinite plate:

$$\vartheta(x/R,\ \mathrm{Fo})=1-\exp\ \Big\{\ln(1-\vartheta_0)-\int\limits_0^{\mathrm{Fo}}\mathrm{Bi}(\mathrm{Fo})\,d\mathrm{Fo}+\frac{1}{6}\ \mathrm{Bi}(\mathrm{Fo})[1-3(x/R)^2]-\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{\mu_n^2}\ \cos\mu_n\frac{x}{R}\exp(-\mu_n^2\,\mathrm{Fo})\times\int\limits_0^{\mathrm{Fo}}\exp(\mu_n^2\,\mathrm{Fo})\,\mathrm{Bi}'(\mathrm{Fo})\,d\mathrm{Fo}\Big\}. \tag{10}$$

Infinite cylinder:

$$\vartheta(r/R,\ \mathrm{Fo}) = 1 - \exp\Big\{\ln(1-\vartheta_0) - 2\int_0^{\mathrm{Fo}} \mathrm{Bi}(\mathrm{Fo})\,d\,\mathrm{Fo} + \frac{1}{4}\,\mathrm{Bi}(\mathrm{Fo})\,[1-2(r/R)^2] - \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 J_0(\mu_n)}\, J_0(\mu_n r/R) \times \exp(-\mu_n^2\,\mathrm{Fo}) \int_0^{\mathrm{Fo}} \exp(\mu_n^2\,\mathrm{Fo})\,\mathrm{Bi}'(\mathrm{Fo})\,d\mathrm{Fo}\Big\}. \tag{11}$$

Sphere:

$$\vartheta(r/R,\ \mathrm{Fo}) = 1 - \exp\Big\{\ln(1-\vartheta_0) - 3\int_0^{\mathrm{Fo}} \mathrm{Bi}(\mathrm{Fo})\,d\mathrm{Fo} + \frac{1}{10}\,\mathrm{Bi}(\mathrm{Fo})\,[3-5(r/R)^2] - \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \cos\mu_n}\, \frac{R\sin\mu_n r/R}{r\mu_n} \times \exp(-\mu_n^2\,\mathrm{Fo}) \int_0^{\mathrm{Fo}} \exp(\mu_n^2\,\mathrm{Fo})\,\mathrm{Bi}'(\mathrm{Fo})\,d\mathrm{Fo}\Big\}. \tag{12}$$

It can be easily seen that the expressions (10)-(12) are exact solutions of Eq. (1) with boundary conditions (2)-(4) at the point $\Psi = 0$.

Temperature Field in an Infinite Plate for $\vartheta_0 = 0.336$

Fo	$\vartheta_{\text{surface}}$			$\vartheta_{\text{center}}$		
	according to (10)	according to the finite-difference method	δ, %	according to (10)	according to the finite-difference method	δ, %
0.5	0.5560	0.5257	—5.83	0.3971	0.4004	—0.827
1.0	0.7143	0.6777	—5.41	0.5691	0.5379	—5.800
1.5	0.8135	0.7693	—5.75	0.7016	0.6619	—6.000
2.0	0.8983	0.8473	—5.99	0.8151	0.7703	—5.810
2.5	0.9275	0.8834	—5.00	0.8779	0.8333	—5.350
3.0	0.9543	0.9210	—3.62	0.9281	0.8781	—5.680
3.5	0.9764	0.9465	—3.15	0.9555	0.9172	—4.180
4.0	0.9902	0.9640	—2.71	0.9822	0.9441	—4.040

Numerical calculations according to [1] show that the error of Eqs. (10)-(12) does not exceed 6% over the whole range of variation of Ψ for $\mathrm{Bi} \le 1.2$ (infinite plate), $\mathrm{Bi} \le 1.35$ (infinite cylinder), or $\mathrm{Bi} \le 1.5$ (sphere). The accuracy of the calculation increases with decreasing Bi.

As an example, we have calculated the heating of an infinite plate when the Biot number varies according to the law $\mathrm{Bi}(\mathrm{Fo}) = 1.2 - \exp(-\mathrm{Fo})$ (cf. table.).

Thus Eqs. (10)-(12) can be used for an approximate calculation of the temperature field when $\mathrm{Bi} = \mathrm{Bi}(\mathrm{Fo})$.

REFERENCE

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